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1991 J. Phys. A: Math. Gen. 24 2889

(http://iopscience.iop.org/0305-4470/24/12/027)

J. Phys. A. Math. Gen. 24 (1991) 2889–2901. Printed in the UK

On the symmetries of relativistic spin-$\frac{1}{2}$ particle equations

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Received 21 January 1991

Abstract. The symmetries of relativistic spin-$\frac{1}{2}$ particle equations are discussed for the cases of non-zero and zero rest masses The Majorana–Weyl equivalence for the physical neutrino is particularly considered.

1. Introduction

Free relativistic spin-$\frac{1}{2}$ particles fall into two categories according to the cancellation or not of their rest masses. For non-zero rest masses their relativistic description is evidently given by the Dirac equation which is, in particular, convenient for the whole set of leptons (e^-, μ^-, τ^-). If we accept the *theoretical* hypothesis of zero rest masses for the neutrinos (ν_e, ν_μ, ν_τ), we also know that the Weyl equation is then the physically *ad hoc* relativistic formulation while everybody has also learned that a two-component Weyl neutrino can be seen as a four-component Majorana one Such generalities [1, 2] point out the specific interest of Dirac, Weyl and (let us say) Majorana equations for the description of spin-$\frac{1}{2}$ (or helicity-$\frac{1}{2}$ for zero rest masses) particles.

The *kinematic* symmetries of such free relativistic wave equations are very well known since the pioneering work of Wigner [3] on the Poincaré group and its irreducible unitary representations [3, 4], completed by the corresponding study of the conformal group [5] of spacetime. These results led to the important respective concepts of *Poincaré and conformal invariances* associated with coordinate transformations in Minkowski spacetime, characterized hereafter by a pseudo-Euclidean metric $G = \mathrm{diag}(1, -1, -1, -1)$ for (spacetime) events $x \equiv (x^\mu) \equiv (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$.

Poincaré and conformal invariances are thus well known and correspond respectively to *10* and *15* symmetries, which can easily be obtained from the classical infinitesimal Lie method [6] when the associated generators are required to form bases of finite-dimensional Lie algebras—the so-called Poincaré and conformal Lie algebras. In this way, we can point out well-established generators associated with spacetime translations (P^μ), Lorentz rotations ($M^{\mu\nu}$) corresponding to spatial rotations (M^{ij}, $i, j = 1, 2, 3$) and Lorentz boosts (M^{0i}) leading to the 10 generators of the Poincaré algebra, completed by dilations (D) and special conformal transformations (C^μ) in order to get the conformal one.

During the last decade, new symmetries have been found on some relativistic wave equations and more particularly on the Dirac equation for non-zero as well as zero rest mass particles. In fact there are two main teams which have considered these problems (also applied to equations with electromagnetic interactions, for example): McLenaghan and his collaborators [7–10] were mainly interested by the corresponding developments on *curved* spacetime while Fushchich and Nikitin [11, 12] were directly polarized on Minkowski (flat) spacetime. Moreover, let us mention that by restricting the Kamran–McLenaghan results [7] to Minkowski spacetime, Durand *et al* [13] have recently recovered in a covariant way the Fushchich–Nikitin results for both $m \neq 0$ and $m=0$ cases in the Dirac equation while Kamran *et al* [14] have classified the complete sets of operators commuting with the $m \neq 0$ Dirac operator, once again in Minkowski spacetime.

These contributions [7, 12] do add symmetries to the current Poincaré or conformal ones but these additional symmetries do not correspond to infinitesimal generators which form a *closed* Lie algebra. Nevertheless they give rise to new conservation laws [11, 12] and have in this way an intrinsic interest. From the above studies, it is clear that in Minkowski spacetime the $m \neq 0$ Dirac equation admits 26 symmetries and that the $m=0$ Dirac equation leads to 52 symmetries. Compared with the respective 10 (plus the identity operator I) and 15 (plus I) usual Poincaré and conformal symmetries, we thus count 15 and 36 respective extra ones which have been listed by Fushchich–Nikitin [11, 12] and Durand *et al* [13]. All the associated generators are *first-order* differential operators as usual but with *matrix* coefficients.

In this paper we want to emphasize the connection between the descriptions of *physical* particles through the corresponding symmetries. Let us remember that the study of a zero rest mass Dirac particle—the so-called Dirac neutrino—does not present actually a physical interest except if we locate ourselves in the Majorana context [15, 16] with 4×4 Dirac matrices. In fact, one of the main problems which has to be solved is to explain how the 52 symmetries of a Dirac neutrino reduce to the 16 of a Weyl neutrino described by the two-component formulation of Lee and Yang [2, 17, 18]. Another important problem is the connection between the symmetries of this last 2×2 formulation and the (equivalent [16]) 4×4 Majorana one. Such information cannot be found in the above-mentioned papers. The content of this paper is thus as follows. *Section 2* will be devoted to a summary concerning the extension of Lie's method when the generators include matrices containing scalar functions depending on the spacetime coordinates. In *section 3*, we will quote some results (already known) for Dirac particles ($m \neq 0$) as well as for Dirac neutrinos ($m=0$) but we will also discuss parities of Dirac matrices and construct superstructures from these developments. *Section 4* will then study the Majorana–Weyl equivalence in terms of such symmetries.

With the previously mentioned metric, we will adopt Bjorken–Drell's conventions [1] and definitions where Greek indices run from 0 to 3 and Latin ones from 1 to 3. Moreover, our units will be chosen according to $c=1$ and $\hbar=1$. Complex conjugation, transposition and Hermitian conjugation of X will be denoted X^*, X^{T} and $X^\dagger \equiv X^{*\mathrm{T}}$ respectively while we will keep the adjoint field $\bar\Psi$ as the one defined by $\Psi^\dagger\gamma^0$ as usual.

2. Extension of Lie's method

Let us consider any wave equation of the form

$$\Delta\Psi(x)=0 \qquad x=(x^\mu) \tag{2.1}$$

where Δ is a linear differential operator acting on a d-multicomponent wavefunction Ψ depending on the spacetime coordinates. Let Q be a generator of the symmetry algebra of (2.1) in the sense that for any solution Ψ, we want to ensure that $Q\Psi$ is once again a solution, i.e.

$$\Delta(Q\Psi)=0. \tag{2.2}$$

A *sufficient* condition is then given by

$$[Q,\Delta]=\lambda(x)\Delta \tag{2.3}$$

where λ is an arbitrary matrix of dimension d including as elements d^2 *scalar* functions depending on the spacetime coordinates when we require the following general form of the generator:

$$Q=\xi^\mu(x)\partial_\mu+c(x) \qquad \partial_\mu\equiv\frac{\partial}{\partial x^\mu}. \tag{2.4}$$

Let us mention as a general remark that λ could be considered as a differential operator but without any supplementary result in our developments. Here ξ^μ ($\mu=0,1,2,3$) and c are also $d\times d$ matrices developed in the basis associated with the d-multicomponent character of the wavefunction Ψ. Such a demand cancels the requirement that the *whole* set of infinitesimal generators realizes a finite-dimensional Lie algebra: it certainly leads to symmetries defined in the sense of (2.2) and (2.3), including as a subset the effectively closing Lie symmetries associated with the Lie algebra

$$[Q^A,Q^B]=\mathrm{i}f^{ABC}Q^C \qquad A,B,C=1,\ldots,p \tag{2.5}$$

but also to additional ones which do not close and are denoted

$$Q^D \qquad D=p+1,\ldots,r \tag{2.6}$$

r being the *largest* number of symmetries admitted by the wave equation (2.1). Such a property illustrates the fact that we are dealing with an extension of Lie's method which has already been exploited in numerous papers already mentioned in the introduction [7–14].

In the following sections we plan to apply such considerations to four types of *relativistic* wave equations with multicomponent wavefunctions characterized by $d=4$ or 2. The first one is the usual *free* Dirac equation ($d=4$, $m\neq0$) describing spin-$\frac{1}{2}$ non-zero rest mass particles and dealing with a Clifford algebra $\mathscr{C}\ell_4$ characterized by (4×4) γ-matrices such that

$$\{\gamma^\mu,\gamma^\nu\}=\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}\mathbb{1}_4 \tag{2.7}$$

for its *covariant* form or in terms of (4×4) α-matrices such that

$$\{\alpha^\mu,\alpha^\nu\}=2\delta^{\mu\nu}\mathbb{1}_4 \qquad \beta\alpha^i=\gamma^i \qquad \alpha^0=\beta \tag{2.8}$$

for its *Hamiltonian* form. The second wave equation is once again the Dirac one ($d=4$) but for zero rest mass particles ($m=0$), called 'Dirac neutrinos', which do not exist in nature: this context is subtended by the same Clifford algebra $\mathscr{C}\ell_4$ for the covariant form of the equation or by a Clifford algebra $\mathscr{C}\ell_3$, which can be characterized by the set of α^is ($i=1,2,3$) according to (2.8) for the Hamiltonian formulation. The third one deals with the so-called 'Majorana neutrinos' described by the preceding case ($d=4$, $m=0$): it can be realized, for example, by specific choices of Dirac matrices (Majorana representations [15]) with reality conditions on the four-component wavefunction. Finally, the fourth wave equation we are considering is the famous Weyl equation ($d=2$, $m=0$) describing two-component *physical* neutrinos [2, 17, 18]; this description is subtended by the Clifford algebra $\mathscr{C}\ell_2$ with four basis elements given in the fundamental representation as the Pauli and identity matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^0 = \mathbb{1}_2. \tag{2.9}$$

As a last comment at this stage, let us recall that there exists [16] an equivalence between the Majorana and Weyl descriptions so that the numbers of corresponding symmetries have to be put in correspondence, as will be shown in section 4 in the first and second quantized contexts.

3. Dirac symmetries for $m=0$ and $m\neq 0$

Let us first consider the zero rest mass case (section 3.1), which corresponds to the largest number of symmetries, and secondly the non-zero rest mass case (section 3.2), which can be enlightened through the preceding information.

3.1. The $m=0$ case

The *largest* r-value already obtained [12, 13] in the spin-$\frac{1}{2}$ context is $r=52$ when the $(d=4, m=0)$ formulation (2.1) is studied according to (2.3) and (2.4). This corresponds to the equation for Dirac neutrinos

$$\Delta_0\Psi_0 \equiv \mathrm{i}\gamma^\mu\partial_\mu\Psi_0 = 0 \tag{3.1}$$

where the subscript 0 refers to the zero mass case. Here the 4×4 γ-matrices satisfy (2.7) and correspond to the irreducible representation of the Clifford algebra $\mathscr{C}\ell_4$ characterized by 16 fundamental elements. Let us immediately insist on the particularly important element γ^5,

$$\gamma^5 = \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \qquad (\gamma^5)^2 = \mathbb{1}_4 \tag{3.2}$$

which evidently anticommutes with the four γs:

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{3.3}$$

It plays an important role as it is noticed that $\mathrm{i}\gamma^5 Q_0$ is still a symmetry operator if Q_0 is an effective symmetry operator for (3.1). Such a property is evidently associated with the *chirality invariance* of this $m=0$ theory. We effectively get

$$[\gamma^5 Q_0, \Delta_0] = \gamma^5\lambda\Delta_0 + 2\gamma^5\Delta_0 Q_0$$

due to the anticommutation of γ^5 and Δ_0. Applied to solutions Ψ_0 we thus obtain

$$\Delta_0(Q_0\Psi_0) = 0 \Rightarrow \Delta_0(\mathrm{i}\gamma^5 Q_0\Psi_0) = 0 \tag{3.4}$$

as required for recognizing the symmetry character of the operator $\mathrm{i}\gamma^5 Q_0$.

By noticing that the sufficient condition (2.3) becomes

$$[Q_0, \Delta_0] = \lambda\Delta_0 \tag{3.5}$$

and leads with the general form (2.4) to the following system:

$$[\xi^\mu(x), \gamma^\nu] + [\xi^\nu(x), \gamma^\mu] = 0 \tag{3.6a}$$

$$[c(x), \gamma^\mu] - \gamma^\nu\partial_\nu\xi^\mu(x) = \lambda(x)\gamma^\mu \tag{3.6b}$$

$$\gamma^\nu\partial_\nu c(x) = 0 \tag{3.6c}$$

we learn after others [12, 13] that 52 parameters are left arbitrary. The corresponding 52 infinitesimal generators can then be summarized as follows: besides the identity $Q_0^1 \equiv \mathbb{1}_4$ we have the 15 conformal generators Q_0^A, $A = 2, 3, \ldots, 16$ realized as usual [5] as

$$P^\mu = i\partial^\mu \qquad M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu + \frac{i}{4}[\gamma^\mu, \gamma^\nu] = L^{\mu\nu} + S^{\mu\nu} \tag{3.7a}$$

$$D = x^\mu P_\mu + \tfrac{3}{2}i \qquad C^\mu = -2x^\mu D + x^\nu x_\nu P^\mu + 2x_\nu S^{\nu\mu} \tag{3.7b}$$

the supplementary $Q_0^{17} \equiv i\gamma^5$ and the corresponding Q_0^B ($B = 18, \ldots, 32$) obtained from the generators (3.7) through a multiplication by $i\gamma^5$, i.e.

$$Q_0^B = i\gamma^5 Q_0^A \tag{3.8}$$

according to (3.4) for example. To these 32 operators we finally have to add the 20 Q_0^D ($D = 33, \ldots, 52$) ones obtained as follows:

$$\omega_0^{ij} = \gamma^i P^j - \gamma^j P^i (\equiv i\gamma^5 \omega_0^{0k}) \qquad \omega_0^{0k} = \gamma^0 P^k - \gamma^k P^0 (\equiv -i\gamma^5 \omega_0^{ij}) \tag{3.9a}$$

$$A_0^\mu = \gamma^\mu - i x_\nu \omega_0^{\mu\nu} \qquad i\gamma^5 A_0^\mu \tag{3.9b}$$

and

$$Z_0^{ij} = [A_0^i, C^j](\equiv -i\gamma^5 [A_0^0, C^k]) \qquad Z_0^{0k} = [A_0^0, C^k](\equiv i\gamma^5 [A_0^i, C^j]) \tag{3.9c}$$

where (i, j, k) do form cyclic permutations of $(1, 2, 3)$ and where the forms given in brackets are only true when they are acting on solutions Ψ_0.

Such an enumeration is easily put in correspondence with the Durand *et al* set [13] of generators issued from Killing, Yano and Penrose–Floyd tensors: the 15 conformal Killing vectors correspond to our generators (3.7), the 15 Yano tensors to (3.8) and the 20 Penrose–Floyd tensors to those given in (3.9).

A remarkable fact is that we can classify these operators in two dual sets of 26 according to the role played by the identity and γ^5 matrices in this $m = 0$ theory admitting the above-mentioned chirality invariance.

Let us now see the γ-matrices defined in agreement with (2.7) as '*fermionic* variables' or as *odd* matrices realized (for an evident connection with odd generators of Lie superalgebras [19]) as off-diagonal matrices. As an example we can choose the so-called '*chiral*' representation with the *odd* γ^μs

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \tag{3.10}$$

and get γ^5 as an interesting (diagonal) *even* matrix given by

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}. \tag{3.11}$$

This distributes the 16 elements of the $\mathscr{C}\ell_4$ algebra in two sets of eight even ($\mathscr{E}$) and eight odd ($\mathscr{O}$) matrices according to their commutation or anticommutation with γ^5, i.e.

$$[\gamma^5, \mathscr{E}] = 0 \qquad \{\gamma^5, \mathscr{O}\} = 0. \tag{3.12}$$

With these characteristics it is straightforward to notice that the previous 52 symmetries fall into a class of 32 *even* ones given by the generators (3.7) and (3.8) besides the identity and γ^5 matrices and characterized by

$$[\gamma^5, Q_0^A] = 0 \qquad A = 1, 2, \ldots, 32 \tag{3.13a}$$

as well as into a class of 20 *odd* generators listed in (3.9) and characterized by

$$\{\gamma^5, Q_0^D\} = 0 \qquad D = 33, \ldots, 52. \tag{3.13b}$$

The even generators close under commutation relations and form an algebra isomorphic to the direct sum $o(4,2) \oplus o(4,2)$. Due to the properties (2.5) and (3.13*a*), we always have

$$\begin{aligned} &[\tfrac{1}{2}(1+\gamma^5)Q_0^A, \tfrac{1}{2}(1-\gamma^5)Q_0^B] = 0 \\ &[\tfrac{1}{2}(1\pm\gamma^5)Q_0^A, \tfrac{1}{2}(1\pm\gamma^5)Q_0^B] = \mathrm{i}f^{ABC}\tfrac{1}{2}(1\pm\gamma^5)Q_0^C \end{aligned} \tag{3.14}$$

so that the above direct sum is trivially obtained as a chiral and orthogonal Lie algebra.

If we are interested in Lie superalgebras [19], we notice that, due to the definite parity of the generators, the Lie bracket requires commutators and anticommutators according to

$$[\mathscr{E}, \mathscr{E}] \sim \mathscr{E} \qquad [\mathscr{E}, \mathscr{O}] \sim \mathscr{O} \qquad \{\mathscr{O}, \mathscr{O}\} \sim \mathscr{E}. \tag{3.15}$$

Among the 52 previous generators, it is not possible to find a subset satisfying (3.15). If we agree to consider the 10 *second*-order operators $P^\mu P^\nu$, then we can determine the following superstructure generated by 29 operators:

$$I \oplus [(P^\mu, M^{\mu\nu}, D) \square (\mathrm{i}\gamma^5, P^\mu P^\nu, \omega_0^{\mu\nu})]. \tag{3.16}$$

We recognize the Weyl algebra in the semi-direct sum ($\square$) with a superalgebra containing 17 ($11\mathscr{E} + 6\mathscr{O}$) operators. This superstructure is a non-simple, non-semi-simple but solvable algebra.

Let us end this section by noticing that an equivalent study can be developed for this ($d = 4$, $m = 0$) case from the Hamiltonian form

$$\mathrm{i}\partial_t \Psi_0 = (\boldsymbol{\alpha} \cdot \boldsymbol{p}) \Psi_0 \tag{3.17}$$

where the 4×4 matrices α^i are implied in a Clifford algebra $\mathscr{C}\ell_3$ characterized by only eight basis elements

$$\mathbb{1}_4, \alpha^1, \alpha^2, \alpha^3, \alpha^1\alpha^2, \alpha^1\alpha^3, \alpha^2\alpha^3, \gamma^5 = -\mathrm{i}\alpha^1\alpha^2\alpha^3. \tag{3.18}$$

Here the differential operator becomes

$$\Delta_0' \equiv \mathrm{i}\partial_t - \boldsymbol{\alpha} \cdot \boldsymbol{p} \tag{3.19}$$

and the associated new problem asking for generators Q_0' is summarized by the equation

$$[Q_0', \Delta_0'] = \mu(x) \Delta_0'. \tag{3.20}$$

We only get the 32 *even* generators Q_0^A ($A = 1, 2, \ldots, 16, 17, \ldots, 32$) from (3.7) and (3.8) as the chirality invariance is still valid. An illuminating fact which explains this ($52 \to 32$) reduction of symmetries is that the Clifford algebra $\mathscr{C}\ell_3$ is just the *even* part of the Clifford algebra $\mathscr{C}\ell_4$ when we attribute the parities as already discussed; indeed, let us remember that the γ-matrices are odd and that the above α-matrices are double products of γs. They are thus even matrices as well as γ^5, which is seen as a product of three αs *or* of four γs.

Such results show the main role played by the basis elements of the corresponding Clifford algebra as it has already been illustrated in supersymmetric quantum mechanics [20]. This will also enlighten the discussion in section 4.

3.2. *The $m \neq 0$ case*

The non-zero rest mass case is described by the original Dirac wave equation written in a covariant way as

$$\Delta_D \Psi_D \equiv (i\gamma^\mu \partial_\mu - m)\Psi_D = 0 \tag{3.21}$$

where once again the 4×4 γ-matrices satisfy (2.7) and form an irreducible representation of the Clifford algebra $\mathscr{C}\ell_4$. The new sufficient condition (2.3) is now

$$[Q, \Delta_D] = \eta(x)\Delta_D \tag{3.22}$$

leading to the new system (compare with (3.6)):

$$[\xi^\mu(x), \gamma^\nu] + [\xi^\nu(x), \gamma^\mu] = 0 \tag{3.23a}$$

$$[c(x), \gamma^\mu] - \gamma^\nu \partial_\nu \xi^\mu(x) = \eta(x)\gamma^\mu \tag{3.23b}$$

$$\gamma^\nu \partial_\nu c(x) = i\eta(x)m. \tag{3.23c}$$

The two first sets of equations ((3.23a) and (3.23b)) are identical to (3.6a) and (3.6b) in the zero-mass case but the last equation (3.23c) is typical of the non-cancellation of the particle mass. Then let us immediately point out that this third equation requires the cancellation of all the scalar functions inside the 4×4 matrix $\eta(x)$. Indeed, from (3.23b) we learn that $c(x)$ and $\eta(x)$ have to be matrices of functions with the same degree in the coordinates while (3.23c) shows that this is impossible *except* if $\eta(x) \equiv 0$. We thus come back to a simpler problem with respect to the one characterized by (3.6). Due to the commutativity of the operator m, we finally get the condition

$$[Q, \Delta_0] = 0 \tag{3.24}$$

leading to the simplified system

$$[\xi^\mu(x), \gamma^\nu] + [\xi^\nu(x), \gamma^\mu] = 0 \tag{3.25a}$$

$$[c(x), \gamma^\mu] - \gamma^\nu \partial_\nu \xi^\mu(x) = 0 \tag{3.25b}$$

$$\gamma^\nu \partial_\nu c(x) = 0. \tag{3.25c}$$

We obtain 26 parameters [12] in comparison with the 52 for the $m = 0$ case. The explicit forms of the generators are given by the (evidently expected) 10 Poincaré generators (3.7a) (plus the identity $\mathbb{1}_4$) supplemented by the 15 following ones:

$$\eta^\mu = i\gamma^5(P^\mu - \gamma^\mu m) \tag{3.26a}$$

$$\omega^{\mu\nu} = \gamma^\mu P^\nu - \gamma^\nu P^\mu - [\gamma^\mu, \gamma^\nu]\frac{m}{2} \tag{3.26b}$$

$$A^\mu = \gamma^\mu - ix_\nu \omega^{\mu\nu} \tag{3.26c}$$

$$B = \gamma^5(x^\mu P_\mu + \tfrac{3}{2}i - \gamma^\mu x_\mu m). \tag{3.26d}$$

This $r = 26$ set shows that $p = 11$ is the largest order of a closed Lie algebra, i.e. the Poincaré algebra (plus the identity). Let us also mention that by eliminating the mass terms with (3.21), we can distinguish 16 *even* generators ($I, P^\mu, M^{\mu\nu}, \eta^\mu, B$) and 10 *odd* ones ($\omega^{\mu\nu}$, A^{μ}) according to the parities previously discussed. Then they take the respective forms ($\varepsilon_{0123}=1$)

$$\eta^{\mu}=\tfrac{1}{2}\varepsilon^{\mu}{}_{\nu\rho\sigma}\gamma^{\nu}\gamma^{\rho}P^{\sigma} \tag{3.27a}$$

$$\omega^{\mu\nu}=\mathrm{i}\gamma^{5}(\gamma^{\mu}P^{\nu}-\gamma^{\nu}P^{\mu})\equiv\mathrm{i}\gamma^{5}\omega_{0}^{\mu\nu} \tag{3.27b}$$

$$A^{\mu}=\gamma^{\mu}+\tfrac{1}{2}\gamma^{5}g^{\mu\lambda}\varepsilon_{\lambda\nu\rho\sigma}x^{\nu}\omega_{0}^{\rho\sigma} \tag{3.27c}$$

$$B=\tfrac{3}{2}\mathrm{i}\gamma^{5}+\frac{\mathrm{i}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}x^{\rho}P^{\sigma} \tag{3.27d}$$

I, P^{μ}, $M^{\mu\nu}$ being unchanged. Once again by considering the 10 *second*-order (even) operators, $P^{\mu}P^{\nu}$ in this $m\neq 0$ case, we can also determine a superstructure (analogous to (3.16) in the $m=0$ case) given as

$$I\oplus[(P^{\mu},M^{\mu\nu})\,\square\,(P^{\mu}P^{\nu},\omega^{\mu\nu})]. \tag{3.28}$$

It is the largest superstructure and it is subtended by 21 *even* operators and *six* odd ones. We recognize the superalgebra already mentioned in the $m=0$ case but the Poincaré algebra replaces the Weyl one as expected and the $\mathrm{i}\gamma^{5}$ symmetry is missing.

As a further comment let us also notice that all the 15 generators (3.26) admit a non-trivial form when the limit $m\to 0$ is considered. We effectively get

$$\begin{aligned}&\eta_{0}^{\mu}=\mathrm{i}\gamma^{5}P^{\mu} \qquad &&\omega_{0}^{\mu\nu}=\gamma^{\mu}P^{\nu}-\gamma^{\nu}P^{\mu}\\ &A_{0}^{\mu}=\gamma^{\mu}-\mathrm{i}x_{\nu}\omega_{0}^{\mu\nu} \qquad &&B_{0}=\gamma^{5}D\end{aligned} \tag{3.29}$$

besides the Poincaré and identity generators (P^{μ}, $M^{\mu\nu}$, I), which, evidently, are still meaningful for light-like unitary representations [3]. Due to the chirality invariance in the $m=0$ case, we thus have to add the following (actually missing) generators:

$$\mathrm{i}\gamma^{5} \qquad \mathrm{i}\gamma^{5}M^{\mu\nu} \qquad \mathrm{i}\gamma^{5}A_{0}^{\mu} \qquad D \qquad C^{\mu} \qquad \mathrm{i}\gamma^{5}C^{\mu} \tag{3.30}$$

where we have included the necessary conformal operators. By commutation relations, we can observe that only commutators between A_{0}^{μ} and C^{μ} are not already included, leading to the supplementary six operators $Z_{0}^{\mu\nu}$ given by (3.9*c*). In this way we have reconstructed the 52 generators of the $m=0$ case as given in section 3.1.

Finally, let us mention that the Hamiltonian form of the Dirac equation could lead us to the same results but with the differential operator

$$\Delta_{\mathrm{D}}'\equiv\mathrm{i}\partial_{t}-\boldsymbol{\alpha}\cdot\boldsymbol{p}-\beta m \tag{3.31}$$

where $\boldsymbol{\alpha}$ and β are the four 4×4 α-matrices ensuring (2.8) of the Clifford algebra $\mathscr{C}\ell_{4}$. The sufficient condition would become here

$$[Q,\Delta_{\mathrm{D}}']=0. \tag{3.32}$$

4. The symmetries of the (physical) neutrino descriptions

As already mentioned in the introduction, the (zero rest mass) physical neutrino is described by a two-component formulation—the Weyl equation—subtended by the Clifford algebra $\mathscr{C}\ell_{2}$ *or* by a four-component formulation—the Majorana equation—written in terms of (4×4) matrices belonging to the Clifford algebra $\mathscr{C}\ell_{3}$ characterizing its Hamiltonian form. Notice that these Clifford algebras are both generated by *three* anticommuting elements, σ_{1}, σ_{2}, σ_{3} in $\mathscr{C}\ell_{2}$ and α_{1}, α_{2}, α_{3} in $\mathscr{C}\ell_{3}$ respectively.

Let us first consider the ($d=2$, $m=0$) Weyl equation for a left-handed neutrino obtained from the ($d=4$, $m=0$) Dirac formulation (*section 4.1*), enhancing the particular role played by the chiral projection operators in this symmetry context. Let us secondly study the ($d=4$, $m=0$) Majorana description (*section 4.2*) in two different matrix representations when only the *first* quantization is considered. Finally (*section 4.3*), let us consider the Majorana–Weyl equivalence [16, 21] by looking at specific symmetry requirements on generators in both contexts, the second quantized level being also considered.

4.1. On the Dirac–Weyl reduction for a left-handed neutrino

Let us recall [2, 22, 23] that, in covariant or Hamiltonian forms, the Weyl equation [24] is characterized by a differential operator of the (3.1) type but where the γ-matrices are essentially replaced by 2×2 σ-matrices. In fact, in terms of the chiral projections of the Dirac wavefunction expressed in the representation (3.10) for example, i.e.

$$\Psi_0(x)=\tfrac{1}{2}(1+\gamma^5)\Psi_0(x)+\tfrac{1}{2}(1-\gamma^5)\Psi_0(x)=\begin{pmatrix}0\\ \eta(x)\end{pmatrix}+\begin{pmatrix}\varphi(x)\\ 0\end{pmatrix} \tag{4.1}$$

we know that the ($d=4$, $m=0$) Dirac system (3.1) decomposes into the two *uncoupled* Weyl equations

$$\sigma^\mu\partial_\mu\eta(x)=0 \qquad \{\sigma^\mu\}\equiv\{\sigma^0\equiv\mathbb{1}_2,\boldsymbol{\sigma}\} \tag{4.2}$$

and

$$\tilde{\sigma}^\mu\partial_\mu\varphi(x)=0 \qquad \{\tilde{\sigma}^\mu\}\equiv\{\sigma^0,-\boldsymbol{\sigma}\} \tag{4.3}$$

where we have used the Carruthers notation tilde [23]. Through plane wave solutions, it is easy to convince ourselves that (4.2) is associated with the description of right-handed (or positive helicity) particles *and* left-handed (or negative helicity) antiparticles while (4.3) is convenient for left-handed particles and right-handed antiparticles. According to the experimental results [25] for electronic neutrinos, which are left-handed, we thus consider (4.3), leading to a system identical with (3.6) but with the replacement of the γ^μ matrices by the $\tilde{\sigma}^\mu$ ones. We effectively get the conditions

$$[\xi^\nu(x),\tilde{\sigma}^\mu]+[\xi^\mu(x),\tilde{\sigma}^\nu]=0 \tag{4.4a}$$

$$[c(x),\tilde{\sigma}^\mu]-\tilde{\sigma}^\nu\partial_\nu\xi^\mu(x)=\rho(x)\tilde{\sigma}^\mu \tag{4.4b}$$

$$\tilde{\sigma}^\mu\partial_\mu c(x)=0. \tag{4.4c}$$

The resolution of such a system shows that there are *16* constants left arbitrary so that we recover the (expected) results of the *conformal invariance* of the Weyl equation: up to the identity, we have the 15 symmetries associated with those explicitly given in (3.7), but here in terms of 2×2 Pauli matrices when needed.

If we remember that the Dirac system (3.5) or (3.6) has led to *52* symmetries, it is interesting to understand how, through the projection procedure (4.1), we can explain the ($52\to16$) reduction in the above considerations. This is a beautiful example which can be treated by a method discussed elsewhere [11]. Indeed, the corresponding problem consists of requiring simultaneously the following two systems:

$$\mathrm{i}\gamma^\mu\partial_\mu\Psi_0(x)=0\equiv\Delta_0\Psi_0(x) \tag{4.5}$$

$$\tfrac{1}{2}(1+\gamma^5)\Psi_0(x)=0\equiv\Delta_1\Psi_0(x) \tag{4.6}$$

the second one requiring (4.3) as the only valuable equation. We are thus searching for the most general symmetry operator $Q_0 \equiv (2.4)$ such that

$$\Delta_0(Q_0\Psi_0) = 0 \qquad \text{and} \qquad \Delta_1(Q_0\Psi_0) = 0. \tag{4.7}$$

This will be determined by the *sufficient* conditions

$$[\Delta_0, Q_0] = \lambda_1\Delta_0 + \lambda_2\Delta_1 \tag{4.8a}$$

$$[\Delta_1, Q_0] = \lambda_3\Delta_0 + \lambda_4\Delta_1 \tag{4.8b}$$

where the four λs are arbitrary. Equation (4.8b) leads to

$$\lambda_3 \equiv 0 \qquad \lambda_4(1+\gamma^5) \equiv 0 \tag{4.9}$$

by noticing that, acting on the solution Ψ_0, the operator Q_0 is easily determined as independent of time derivatives while Δ_0 is explicitly dependent so that the arbitrary operator λ_3 has to become zero. This implies from (4.8b) and (4.6) that the operator $\lambda_4(1+\gamma_5)$ has also to become zero, leading finally to the condition

$$[Q_0, \gamma^5] = 0. \tag{4.10}$$

Then an analogous method applied to (4.8a) leads to the supplementary simplification

$$\lambda_2(1+\gamma^5) \equiv 0. \tag{4.11}$$

We see that the present problem comes back to the form (3.5) with $\lambda = \lambda_1$ when we, moreover, require (4.10). It is trivial to show that, among the 52 generators (3.7)–(3.9), only 32 of them commute with γ^5. Moreover, they fall into two *equivalent* sets of 16 operators due to immediate properties such that

$$\tfrac{1}{2}(1-\gamma^5)Q_0 = -\tfrac{1}{2}(1-\gamma^5)\gamma^5 Q_0 \tag{4.12}$$

and by using the condition (4.6) (showing in particular that Ψ_0 and $\gamma^5\Psi_0$ are equivalent solutions up to a sign).

In conclusion, we actually understand why and how the Weyl neutrino admits only *16* symmetries corresponding to the conformal invariance of its wave equation. No extra symmetries issued from the ($d=4$, $m=0$) Dirac developments survive in this physical context.

4.2. *On the Dirac–Majorana description*

The ($d=4$, $m=0$) Dirac description immediately leads to the Majorana one if we require that the matrices are Majorana ones (all the γ^μs have to be imaginary within our conventions so that the α's are purely real and $\alpha^0 \equiv \beta$ purely imaginary (see (2.7) and (2.8)) and if, moreover, we require that the four-component wavefunction is real. A more general way for constructing the Majorana theory [21, 22] is to require (independently of choices of representations) that

$$\Psi(x) = \Psi^C(x) = C^{-1}\bar{\Psi}^T(x) \qquad \bar{\Psi}(x) = \Psi^\dagger(x)\gamma^0 \tag{4.13}$$

where $\bar{\Psi}$ is the adjoint Dirac spinor and C is the matrix (operator) associated with particle–antiparticle or 'charge' conjugation. Let us recall the well-known properties

$$C\gamma^\mu C^{-1} = -\gamma^{\mu^T} \qquad C^\dagger C = I \qquad C^T = -C \tag{4.14}$$

and notice that, in the *chiral* representation (3.10), we get, for example,

$$C = -\gamma^0\gamma^2 \qquad \Psi = \Psi^C = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ i\Psi_2^* \\ -i\Psi_1^* \end{pmatrix} = \begin{pmatrix} \varphi \\ -\sigma^2\varphi^* \end{pmatrix} \tag{4.15}$$

while, in the following *Majorana* representation,

$$\gamma^0 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & -i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix} \tag{4.16}$$

we have

$$C = -\gamma^0 \qquad \Psi = \Psi^C = \Psi^*. \tag{4.17}$$

In both cases we can see that the corresponding *Majorana* equations written in covariant (or Hamiltonian) form lead to 52 (or 32) symmetries according to the results obtained in section 3.1, depending essentially on the corresponding Clifford algebra $\mathcal{C}\ell_4$ ($\mathcal{C}\ell_3$).

4.3. *On the Majorana–Weyl equivalence*

Let us consider the Hamiltonian form of (4.3), i.e.

$$i\partial_t\varphi(x) + \boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi(x) = 0 \equiv L\varphi(x). \tag{4.18}$$

We know [16] that this equation and its complex conjugate lead to the Majorana equation characterized by

$$\varphi(x) = \varphi_R(x) + i\varphi_I(x) \qquad \Psi(x) = \begin{pmatrix} \varphi_R(x) \\ \varphi_I(x) \end{pmatrix} = \Psi^C(x) = \Psi^*(x) \tag{4.19}$$

according to (4.17). The corresponding Majorana equation reads

$$i\partial_t\Psi(x) - \boldsymbol{\alpha}\cdot\boldsymbol{p}\Psi(x) = 0 \equiv \mathcal{L}\Psi(x) \tag{4.20}$$

where

$$\mathcal{L} \equiv \tfrac{1}{2}\begin{pmatrix} L - L^* & i(L + L^*) \\ -i(L + L^*) & L - L^* \end{pmatrix}. \tag{4.21}$$

Expressed in terms of 2×2 block matrices, a symmetry operator Q ensuring that $Q\Psi$ is also a solution of (4.20) takes the explicit form

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \tag{4.22}$$

with

$$Q_1 = Q_4 \qquad Q_2 = -Q_3 \tag{4.23a}$$

and

$$Q_1^* - iQ_3^* = Q_1 - iQ_3 \tag{4.23b}$$

when we require that the corresponding submatrices are symmetry operators of the Weyl equation (4.18) and of its complex conjugate. Conversely, if we require that X is a symmetry operator of the Weyl equation, we will get a symmetry operator of the Majorana equation if

$$Q=\begin{pmatrix} X+X^* & \mathrm{i}(X-X^*) \\ -\mathrm{i}(X-X^*) & X+X^* \end{pmatrix} \tag{4.24}$$

in complete agreement with the conditions (4.23). Through such simple arguments we can easily explain the doubling of the symmetries between the Weyl and Majorana theories. In particular the identity $\mathbb{I}_4$ and the matrix γ^5 in the abbreviated theory correspond to the same symmetry (the identity) in the Weyl context. Moreover, as an exercise we can propose to take one of the (20) supplementary generators with respect to the (32) even ones (i.e. one of those given in (3.9) expressed in the representation (4.16)) and to show that it does not correspond to a Weyl symmetry (i.e. it does not satisfy (4.24)). As a conclusion the 32 Majorana symmetries deduced from the $\mathscr{C}\ell_3$ algebra fall into two equivalent sets of 16 Weyl symmetries, exploiting at this stage the Majorana–Weyl equivalence at the level of unquantized wavefunctions.

Let us end this section by considering the second quantized field context where Dirac, Majorana and Weyl theories are developed in terms of creation and annihilation operators. Let us now deal with Majorana four-component and Weyl two-component fields as anticommutating (Fermi) field operators. Here we *first* point out [22] that the Dirac field Ψ_0 and its charge conjugate $\Psi_0^{\mathrm{C}}=C^{-1}\bar{\Psi}_0^{\mathrm{T}}$ are two particular solutions of the quantized equation (3.1) so that the combination

$$\Psi_0'=\frac{1}{\sqrt{2}}(\Psi_0-\gamma^5C^{-1}\bar{\Psi}^{\mathrm{T}}) \tag{4.25}$$

is still a solution when the γ^5 invariance is once again exploited. Through the identification of the Dirac operator

$$\Psi_0'=U\Psi_0U^{-1} \qquad U^\dagger=U^{-1} \tag{4.26}$$

and the Weyl operator

$$\varphi=\tfrac{1}{2}(1-\gamma^5)\Psi_0 \tag{4.27}$$

we know that [22]

$$\begin{aligned} U\varphi U^{-1}&=\tfrac{1}{2}(1-\gamma^5)\left[\frac{1}{\sqrt{2}}(\Psi_0+C^{-1}\bar{\Psi}_0^{\mathrm{T}})\right] \\ &=\tfrac{1}{2}(1-\gamma^5)\chi \end{aligned} \tag{4.28}$$

and

$$\chi=C^{-1}\bar{\chi}^{\mathrm{T}}=\chi^{\mathrm{C}} \tag{4.29}$$

showing that we have constructed a Majorana field χ (4.29) whose projection is unitarily equivalent to a Weyl field. Once again we see here, in this second quantized context, that χ and $\gamma^5\chi$ are *two* different Majorana fields which correspond to the same Weyl field, another way of explaining the doubling already noticed at the first quantized level.

Acknowledgments

Interesting discussions with J Serpe on his paper concerning the Majorana–Weyl equivalence and with N Kamran (in Montréal), W Fushchich and A Nikitin (in Moscow) on their respective approaches to the subject are acknowledged.

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